Premultiplying Eq. (8) by $\underline{\phi}_j^T$ $(j \neq i)$ and applying orthogonality

$$\alpha_{i,j} = \frac{\underline{\phi}_j^T \underline{Q} \underline{\phi}_i}{(\lambda_i - \lambda_j)\underline{\phi}_j^T \underline{A} \underline{\phi}_j} \tag{14}$$

Similarly, from Eq. (9) by $\underline{\phi}_i^T$

$$\zeta_i = \frac{\underline{\phi}_i^T \underline{Q} \underline{\mu}_i}{\underline{\phi}_i^T \underline{A} \underline{\phi}_i} \tag{15}$$

and by $\underline{\phi}_j^T$

$$\beta_{i,j} = \frac{1}{(\lambda_i - \lambda_j)\underline{\phi}_j^T \underline{A} \underline{\phi}_j} \left\{ \underline{\phi}_j^T \underline{Q} \underline{\mu}_i - \frac{\underline{\phi}_j^T \underline{Q} \underline{\phi}_i \cdot \underline{\phi}_i^T \underline{Q} \underline{\phi}_i}{(\lambda_i - \lambda_j)\underline{\phi}_i^T \underline{A} \underline{Q}_i} \right\} \tag{16}$$

Transformation to Different Sensitivities

The relationship between this perturbation method and the direct differential methods now will be clarified.

The variable ϵ may be equated to an increment δv_k in the kth design variable V_k. Consequently the matrix Q becomes equal to the variation in $(\lambda \underline{A} + \underline{B})$ due to a unit change in V_k.

We may then compare the power series (4) and (5) with the corresponding Taylor series.

$$\lambda_i^p = \lambda_i + \delta V_k \frac{\partial \lambda_i}{\partial V_k} + \frac{(\partial V_k)^2}{2} \frac{\partial^2 \lambda_i}{\partial V_k^2} + \ldots \tag{17}$$

$$\underline{\phi}_i^p = \underline{\phi}_i + \delta V_k \frac{\partial \underline{\phi}_i}{\partial V_k} + \frac{(\delta V_k)^2}{2} \frac{\partial^2 \underline{\phi}_i}{\partial V_k^2} + \ldots \tag{18}$$

giving

$$\frac{\partial \lambda_i}{\partial V_k} = \eta_i \tag{19}$$

$$\frac{\partial \underline{\phi}_i}{\partial V_k} = \underline{\mu}_i \tag{20}$$

These results coincide with those given by Refs. 1 and 3.

Further

$$\frac{\partial^2 \lambda_i}{\partial V_k^2} = 2\zeta_i \tag{21}$$

$$\frac{\partial^2 \underline{\phi}_i}{\partial V_k^2} = 2\nu_i \tag{22}$$

the results which were to be established here.

Physical Significance of Higher-Order Sensitivities

The authors previously cited have remarked at the intuitive appeal of the first-order sensitivities. The sensitivity of frequency (or λ) can be computed solely on the basis of knowledge of the associated eigenvector. The eigenvector sensitivity can be computed similarly using data which are natural products of previous analyses. It should be noted, however, that the term $(\lambda_i - \lambda_j)$ in the denominator of Eq. (14) shows that numerical difficulties may be encountered for relative sensitivities of eigenvectors with close natural frequencies. The equation will degenerate for $\lambda_i = \lambda_j$ and Eq. (14) cannot be used. The paper by Lancaster provides analyses which may be useful if this occurs.

Turning now to the second-order sensitivities and discussing only the frequency sensitivity, it can be seen that the second-order sensitivity of the eigenvalue involves the first-order sensitivity of the eigenvector. Thus, if the first eigenvector sensitivity is large (poorly conditioned) due to close eigenvalues, then the second eigenvalue sensitivity will be large.

The second-order eigenvector sensitivity involves $\beta_{i,j}$ from Eq. (16). In this case, the frequency difference is represented by $(\lambda_i - \lambda_j)^2$ in the denominator. Hence, whatever condition (good or bad) is present in the second eigenvalue sensitivity will be exaggerated in the second eigenvector sensitivity.

Conclusions

The extrapolation of first-order sensitivities for gross changes is dependent on the size of the second-order sensitivity (which must be negligible).

The second-order sensitivity of the eigenvalue may be computed from knowledge of the first-order sensitivity of eigenvectors. A necessary (although not sufficient) condition for this second-order eigenvalue sensitivity to be negligible is the sufficient separation of the chosen eigenvalues from all other eigenvalues.

Acknowledgments

The author is grateful for the support and encouragement of his supervisor Dr. A. Cowley. This work is part of a course of study funded by the Science and Engineering Research Council.

References

[1] Fox, R. L. and Kapoor, M. P., "Rate of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, Dec. 1968, pp. 2426-2429.

[2] Garg, S., "Derivatives of Eigensolutions for a General Matrix," *AIAA Journal*, Vol. 11, Aug. 1973, pp. 1191-1194.

[3] Plaut, R. H. and Huseyin, K., "Derivatives of Eigenvalues and Eigenvectors in Non-Self-Adjoint Systems," *AIAA Journal*, Vol. 11, Feb. 1973, pp. 250-251.

[4] Nelson, R. B., "Simplified Calculation of Eigenvector Derivative," *AIAA Journal*, Vol. 14, Sept. 1976, pp. 120-125.

[5] Vanhonacker, P., "An Introduction to Sensitivity Analysis," Sixth Modal Analysis Seminar, University of Leuven, Sept. 1981.

[6] Lancaster, P., "Free Vibrations of Lightly Damped Systems by Perturbation Methods," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. XIII, Pt. 2, 1960, pp. 138-155.

Dynamic Condensation

Mario Paz*
University of Louisville, Louisville, Kentucky

Introduction

THE problem of determining a sufficient number of natural frequencies and their corresponding modal shapes to define the dynamic characteristics of the structure is very important and fundamental in structural dynamics. The solution of the corresponding eigenproblem can be obtained readily for structures modeled with relatively few or a moderate number of degrees of freedom. However, the mathematical formulation to accurately describe the structure usually requires a discretization process with a large number of elements and nodes or joints; and, therefore, it results in a large number of degrees of freedom. In this case, a reduction of the number of independent degrees of freedom is often required before proceeding to the calculation of the required number of natural frequencies and modal shapes.

Received April 9, 1983; revision received May 17, 1983.

*Professor and Chairman, Civil Engineering Department.

A simple method commonly used to accomplish this reduction is known as the static condensation method proposed by Guyan[1] and also by Irons[2] almost 20 years ago. Guyan method of reduction, although exact for the reduction of static problems, introduces large errors when applied to the reduction of dynamic problems. In an attempt to decrease these errors, many investigators[3-12] have proposed and discussed modifications of the original Guyan method. However, the proposed modifications are based on a truncated series expansion and, therefore, depend on the convergence of the series. In addition, the calculation of the terms left in the series requires the evaluation of the inverse of a large matrix. The various methods proposed as modifications of Guyan reduction, generally are, intended to improve the eigenvectors, although not the eigenvalues. These methods depend directly on the original Guyan method for static condensation and, as a consequence, the improved eigensolution provides acceptable values for only a few of the lower modes left in the reduced problem.

Algorithm for Dynamic Condensation

The algorithm of the proposed method[13] may be explained by considering the generalized eigenvalue problem expressed in partitioned matrices as follows:

$$\begin{bmatrix} [K_{ss}]-\omega^2[M_{ss}] & [K_{sp}]-\omega^2[M_{sp}] \\ [K_{ps}]-\omega^2[M_{ps}] & [K_{pp}]-\omega^2[M_{pp}] \end{bmatrix} \begin{bmatrix} \{Y_s\} \\ \{Y_p\} \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{0\} \end{bmatrix} \tag{1}$$

where $[K]$ and $[M]$ are, respectively, the stiffness and mass matrices; $\{Y_s\}$ the displacement vector corresponding to the s degrees of freedom to be reduced and $\{Y_p\}$ the vector corresponding to the remaining p independent degrees of freedom; and ω^2 is the eigenvalue determined approximately at each step of the calculations.

The process starts by assigning to ω^2 any approximate or zero value for the first eigenvalue; it is followed by elementary operations applied to eliminate the first s displacement in Eq. (1). At this state of the elimination process Eq. (1) may be written as

$$\begin{bmatrix} [I] & [-\bar{T}] \\ [0] & [\bar{D}] \end{bmatrix} \begin{bmatrix} \{Y_s\} \\ \{Y_p\} \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{0\} \end{bmatrix} \tag{2}$$

in which $[\bar{T}]$ is the transformation matrix defined in the following relation

$$\{Y\} = \begin{Bmatrix} \{Y_s\} \\ \{Y_p\} \end{Bmatrix} = \begin{Bmatrix} [\bar{T}] \\ [I] \end{Bmatrix} \{Y_p\} \tag{3}$$

and $[\bar{D}]$ the dynamic matrix which satisfies the relation

$$[\bar{K}] = [\bar{D}] + \omega^2[\bar{M}] \tag{4}$$

where $[\bar{K}]$ and $[\bar{M}]$ are, respectively, the stiffness and mass matrices of the reduced system. The reduced mass matrix is calculated from

$$[\bar{M}] = [T]^T[M][T] \tag{5}$$

where

$$[T] = \begin{bmatrix} [\bar{T}] \\ [I] \end{bmatrix} \tag{6}$$

while the reduced stiffness matrix is determined using Eq. (4).

The resulting reduced eigenproblem

$$[[\bar{K}] - \omega^2[\bar{M}]]\{Y_p\} = \{0\} \tag{7}$$

is solved to obtain a virtually exact eigenvalue and corresponding eigenvector for the first mode, and an approximate value for the second eigenvalue. The calculations continue by substituting into Eq. (1) this approximate value

Table 1 Eigenvalues for a 48-degree-of-freedom system reduced to four coordinates

		Method of condensation			
		Static	Modified	Dynamic	Dynamic (one iteration)
Mode	Exact eigenvalue	Eigenvalue (% error)			
1	0.6168	0.6244 (1.2)	0.6244 (1.2)	0.6244 (1.2)	0.6168 (0.0)
2	5.5478	6.1754 (11.3)	6.1754 (11.3)	5.5565 (0.2)	5.5478 (0.0)
3	15.389	20.063 (30.4)	20.063 (30.4)	15.494 (0.7)	15.389 (0.0)
4	30.099	41.182 (36.8)	41.182 (36.8)	31.159 (3.5)	30.016 (0.2)

Table 2 Maximum displacements for a 48-degree-of-freedom system reduced to four coordinates

		Method of condensation			
Coordinate[a]	Exact displacement	Static	Modified	Dynamic	Dynamic (one iteration)
		Displacement (% error)			
8	21.888	20.753 (5.2)	21.234 (3.0)	22.087 (0.9)	21.930 (0.2)
16	37.852	36.858 (2.6)	37.454 (1.1)	37.736 (0.3)	37.864 (0.0)
24	49.820	50.463 (1.3)	49.629 (0.4)	50.285 (0.9)	49.811 (0.0)
32	59.431	58.935 (0.8)	59.271 (0.3)	59.164 (0.4)	59.441 (0.0)
40	67.068	66.449 (0.9)	66.915 (0.2)	66.705 (0.5)	67.072 (0.0)
48	70.904	71.810 (1.3)	70.626 (0.4)	71.597 (0.7)	70.891 (0.0)
Maximum % error		(12.3)	(6.0)	(3.8)	(0.7)
Average % error		(2.5)	(1.3)	(0.8)	(0.1)

[a] Only six coordinates are listed to shorten the table.

Table 3 Maximum shear forces in the columns of a 48-story shear building reduced to four coordinates

Coordinate[a]	Exact shear force	Method of condensation: Static	Modified	Dynamic	Dynamic (one iteration)
		Shear force (% error)			
8	1460.0	1497.2 (2.6)	1499.5 (2.7)	1473.5 (0.9)	1460.2 (0.0)
16	1287.1	1270.2 (1.3)	1307.7 (1.6)	1320.6 (2.6)	1292.6 (0.4)
24	1227.8	1215.5 (1.0)	1264.8 (3.0)	1279.1 (4.2)	1234.4 (0.5)
32	1196.3	1219.7 (2.0)	1276.4 (6.7)	1208.0 (1.0)	1197.4 (0.1)
40	1036.2	1021.0 (1.5)	1074.5 (3.7)	1039.0 (0.3)	1037.8 (0.2)
48	1604.7	1823.1 (13.6)	1776.4 (10.7)	1540.7 (4.0)	1599.3 (0.3)
Maximum % error		(13.6)	(10.7)	(4.6)	(0.7)
Average % error		(1.9)	(4.1)	(2.2)	(0.3)

[a] Only six coordinates are listed to shorten the table.

for the second eigenvalue followed by application of Eqs. (2-7) to obtain now a virtually exact second eigenvalue and an approximation to the third eigenvalue. The continuation of this process of solution will result in the virtually exact eigensolution for the lowest p eigenvalues and corresponding p eigenvectors of the original system.

In applying the dynamic condensation method to reduce dynamic systems, it has been found that the iterative application of the algorithm for each mode converges rapidly to the exact eigensolution for all the modes left in the reduced system. It has been shown that an eigenvalue calculated for any mode in the iteration process should converge monotonically from above to the exact eigenvalue. This fact may be used to detect any poor choice of primary coordinates which results in a nonconvergent solution.

Numerical Example

The proposed method for dynamic condensation has been applied successfully to several structures for which the exact solution has also been obtained. For comparison purposes, the eigensolution for these structures has also been calculated applying the static condensation method and the modified condensation method used recently by Miller.[14,15] One of the structures analyzed consisted of a uniform 48-story shear building having equal masses at the floor levels which was modeled as a 48 degrees-of-freedom system. This structure was analyzed first with 48 degrees of freedom in free vibration and also for the seismic excitation produced by the design spectrum recommended by Newmark.[16] The results of this analysis, which were considered exact, were used to calculate the errors obtained when the system was solved after a reduction from 48 to 4 coordinates (Y_{12}, Y_{24}, Y_{36}, Y_{48}) by the following methods: 1) static condensation, 2) modified condensation, 3) dynamic condensation, and 4) dynamic condensation with one iteration at each step.

Tables 1, 2, and 3 show the results obtained, respectively, for the four lowest eigenvalues of the system, for the maximum displacements at the level of the floors, and for the maximum shear forces in the columns of the building. In Tables 2 and 3 the "exact" values for the maximum displacement and for the maximum shear force were calculated as the square root of the sum of the squares of the first four modal contributions.

Results shown in Table 1 for this example of 48 degrees of freedom reduced to just four coordinates, indicate the following.

1) Static or modified condensation produced only the fundamental eigenvalue with an acceptable value. Higher eigenvalues have errors in excess of 10% and, therefore, are unacceptable.

2) Dynamic condensation with no iteration gave virtually exact eigenvalues for all the modes left in the reduced problem and with one iteration giving results which may be considered exact.

The comparison of displacements for the 48-story shear building shown in Table 2 indicates that, for this example, static or modified condensation results in maximum error of 12.3% and 6%, respectively, while results of dynamic condensation method are virtually exact with just a single iteration.

Study of the results shown in Table 3 indicate the following.

1) The shear force in the columns of the building determined after using static or modified condensation has maximum errors in excess of 10%.

2) The application of dynamic condensation method results in a relatively small error (4.6% maximum error).

3) The use of dynamic condensation with one iteration gives for this example virtually exact shear force values.

Conclusions

A method based on dynamic condensation which requires only elementary operations as used in the solution of a linear system of equation has been proposed for the reduction of dynamic systems. The application of the proposed method to structural problems gives virtually exact eigenvalues and eigenvectors for all the modes considered in the reduced eigenproblem. The proposed algorithm is applied progressively from the fundamental mode to any desired number of higher modes. Though not necessary, an iterative process may be implemented in the application of the dynamic condensation method to improve further the solution. In this case the use of one or two iterative cycles is sufficient to converge to the exact solution for all the modes left in the reduced eigenproblem. The dynamic condensation method may also be used to determine resonant frequencies in a range of interest. The proposed method also permits the detection of a poor selection of primary coordinates for which the process does not converge to the exact solution.

References

[1] Guyan, R. J., "Reduction of Stiffness and Mass Matrices," *AIAA Journal*, Vol. 3, Feb. 1965, p. 380.

[2] Irons, B. M., "Structural Eigenvalue Problems: Elimination of Unwanted Variables," *AIAA Journal*, Vol. 3, May 1965, pp. 961-962.

[3] Anderson, R. G., Irons, B. M., and Zienkiewicz, O. C., "Vibration and Stability of Plates Using Finite Elements," *International Journal of Solids and Structures*, Vol. 4, May 1968, pp. 1031-1055.

[4] Wright, G. C. and Miles, G. A., "An Economical Method for Determining the Smallest Eigenvalues of Large Linear Systems,"

International Journal for Numerical Methods in Engineering, Vol. 3, Jan.-March 1971, pp. 25-33.

[5]Bathe, K. J. and Wilson, E. L., "Solution Method for Eigenvalue Problems in Structural Dynamics," *International Journal for Numerical Methods in Engineering*, Vol. 6, No. 2, 1973, pp. 213-226.

[6]Jennings, A., "Mass Condensation and Simultaneous Iteration for Vibration Problems," *International Journal for Numerical Methods in Engineering*, Vol. 6, No. 4, 1973, pp. 543-552.

[7]Kidder, R. L., "Reduction of Structural Frequency Equations," *AIAA Journal*, Vol. 11, June 1973, p. 892.

[8]Kidder, R. L., "Reply by Author to A. H. Flax," *AIAA Journal*, Vol. 13, May 1975, pp. 702-703.

[9]Flax, A. H., "Comment on 'Reduction of Structural Frequency Equations,'" *AIAA Journal*, Vol. 13, May 1975, pp. 701-702.

[10]Hensell, R. D. and Ong, J. H., "Automatic Masters for Eigenvalue Economization," *Earthquake Engineering and Structural Dynamics*, Vol. 3, April 1975, pp. 375-383.

[11]Irons, B. M., "Discussion on 'Dynamic Reduction of Structural Models,'" *Journal of Structural Division ASCE*, Vol. 107, May 1981, pp. 1023-1024.

[12]Flax, A. H., "Discussion on 'Dynamic Reduction of Structural Models,'" *Journal of Structural Division ASCE*, Vol. 107, July 1981, pp. 1393-1394.

[13]Malpartida, C., "Dynamic Condensation of Structural Eigenproblems," M.S. Thesis, Speed Scientific School, University of Louisville, Louisville, Ky., April 1983.

[14]Miller, C. A, "Dynamic Reduction of Structural Models," *Journal of Structural Division ASCE*, Vol. 106, Oct. 1980, pp. 2097-2108.

[15]Miller, C. A., "Errors Resulting from Dynamic Reduction," *Proceedings of the First National Conference on Computer in Civil Engineering*, May 1981, pp. 58-71.

[16]Newmark, N. M. and Hall, W. J., "Procedures and Criteria for Earthquake Resistant Design," *Building Practice for Disaster Mitigation*, Dept. of Commerce, 1973, pp. 209-236.

Thermally Developing Laminar Flow in a Duct with External Radiation and Convection

L. T. Yeh*
Texas Instruments Inc., Lewisville, Texas
and
B. T. F. Chung†
The University of Akron, Akron, Ohio

Nomenclature

B_i = Biot number, $\frac{1}{2}Nu_o$
h_{xi} = local internal heat-transfer coefficient
h_o = external heat-transfer coefficient
k_i = thermal conductivity of internal flow
k_o = thermal conductivity of external flow
m = $(n+1)/n$
n = flow-type parameter, $n=1$ for Newtonian flow
Nr = radiation parameter, $\sigma\epsilon Tr^3R/k_o$
Nu_o = Nusselt number, $2h_oR/k_o$
Pe = Peclet number of inner fluid, $2\bar{u}R/\alpha$
q_o'' = external heat flux
Q_o = dimensionless external heat flux, $q_o''R/k_oT_r$
r = radial coordinate or direction normal to axial axis
R = radius for circular conduit or half-width for flat channel
S = geometry parameter, $S=0$ for flat channel, $S=1$ for pipe
T = temperature
u = velocity of the internal or inner fluid
$\bar{u}$ = mean velocity of inner fluid
x = axial coordinate
X = dimensionless axial coordinate, $(x/R)/Pe$
y = distance from inner wall
α = thermal diffusivity
δ = thermal boundary-layer thickness
ϵ = emissivity of outer wall
σ = Stefan-Boltzmann constant

Subscripts
i = inlet condition for internal flow
o = external surface
r = reference point
w = wall condition
x = axial dependent
∞ = external environment or ambient condition

Introduction

THE classical problem of laminar thermal entrance flow in a circular tube is that in which either the wall temperature or the heat flux at the wall is prescribed.[1] Even in a more complicated case that involves convective heat exchange between the outer surface of the conduit and a fluid environment, it is usually assumed that the value of the external heat-transfer coefficient is known a priori and is normally taken to be a constant. In reality, the external heat-transfer coefficient often depends on the wall temperature, which is an unknown.

The problem of laminar forced convection in pipe flow subjected to thermal radiation has received considerable attention.[2-6] The majority of previous investigations on this subject is limited to the case with the environment temperature at absolute zero. More recently, Faghri and Sparrow[7] applied a numerical scheme developed earlier by Patankar and Spalding to solve the problem of laminar flow in a horizontal pipe subjected to external natural convection and radiation.

This Note considers the same problem presented in Ref. 7, but uses a simpler approach, the heat balance integral method. However, the present analysis is applied equally to the power law fluids (non-Newtonian flow), which none of the previous investigators has considered. Flow inside a flat conduit is also included in this analysis.

Formulation of the Problem

Consideration is given to a laminar flow with constant physical properties and with a uniform temperature T_i entering a channel in which the flow is fully developed hydrodynamically but is developing thermally. It is assumed that the heat is transferred from the inside channel surface by convection and conduction to the fluid and from the outside channel wall by radiation and forced or natural convection to an external environment at temperature T_∞. In addition, the channel is subjected to an external uniform heat flux or an internal uniform generation within the wall. If the viscous dissipation and the wall resistance are negligible, the steady-state temperature field can be described by the following mathematical expression:

$$\frac{1}{r^S}\frac{\partial}{\partial r}\left(r^S\frac{\partial T}{\partial r}\right)=\frac{u}{\alpha}\frac{\partial T}{\partial x} \tag{1}$$

Presented as Paper 83-0529 at the AIAA 21st Aerospace Sciences Meeting, Reno, Nev., Jan. 10-13, 1983; submitted April 9, 1983; revision submitted July 19, 1983.

*Thermal Design Section.
†Professor, Department of Mechanical Engineering.